

BOUNDED STEP FUNCTIONS AND FACTORIAL RATIO SEQUENCES

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ABSTRACT. We study certain step functions whose nonnegativity is related to the integrality of sequences of ratios of factorial products. In particular, we obtain a lower bound for the mean square of such step functions which allows us to give a restriction on when such a factorial ratio sequence can be integral. Additionally, we note that this work has applications to the classification of cyclic quotient singularities.

1. INTRODUCTION

Let u_n be the factorial ratio sequence

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)! (a_2 n)! \cdots (a_K n)!}{(b_1 n)! (b_2 n)! \cdots (b_L n)!},$$

which depends on integer parameters $a_1, \dots, a_K, b_1, \dots, b_L \in \{1, 2, 3, \dots\}$. We consider the classification of parameters $\mathbf{a} = \{a_1, \dots, a_K\}$ and $\mathbf{b} = \{b_1, \dots, b_L\}$ such that u_n is an integer for all $n > 0$. It turns out that this is equivalent to studying the question of when the step function

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

is always nonnegative. In many interesting cases it is natural to restrict to \mathbf{a} and \mathbf{b} with

$$\sum_{k=1}^K a_k = \sum_{l=1}^L b_l,$$

so that $f(x; \mathbf{a}, \mathbf{b})$ is a periodic function with period 1, and so that $u_n(\mathbf{a}, \mathbf{b})$ does not grow too fast. For example, if these sums are equal and $u_n(\mathbf{a}, \mathbf{b})$ is always an integer, then \mathbf{a} and \mathbf{b} can easily be used to give explicit elementary bounds on the prime counting function $\pi(x)$ via a standard method (see [9, Section 5.1], for example).

Two natural parameters associated with any such step function are the difference $h(L, K) = L - K$, which we will call the *height*, and the sum $l(L, K) = L + K$, which we will call the *length*.

If we restrict to the case of height 1, then the situation is well understood (see [1], [6], [11]). Specifically, the set of pairs of tuples (\mathbf{a}, \mathbf{b}) satisfying

$$(1) \quad \sum_{k=1}^K a_k = \sum_{l=1}^{K+1} b_l,$$

Date: March 27, 2008.

$$(2) \quad \gcd(a_1, a_2, \dots, a_K, b_1, b_2, \dots, b_{K+1}) = 1,$$

and

$$(3) \quad a_k \neq b_l \text{ for all } k, l,$$

and such that $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all n is exactly known. There are three easily described two parameter infinite families of such sequences and 52 “sporadic” sequences which are easily listed.

A curious feature of this classification is that there are exactly two parameter sets of length 9 and none larger. It would be nice to have a “simple” reason to explain the fact that a function of the form

$$f(x) = \sum_{k=1}^5 \lfloor a_k x \rfloor - \sum_{l=1}^6 \lfloor b_l x \rfloor$$

cannot be positive for all x if $\sum a_k = \sum b_l$ and $a_k \neq b_l$ for all k, l . Although the classification in [1] proves that this is true, it does not seem to shed any light on exactly *why* this is true.

More generally, if we fix the height $h(L, K)$ we may ask whether there is a reason that the length $l(L, K)$ must be small (in terms of the height) in order for $f(x)$ to have a chance of being always nonnegative. We obtain here a result of this form as follows.

Theorem 1.1. *Fix $L - K = D$. If*

$$(4) \quad f(x) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

where

$$\sum_{k=1}^K a_k = \sum_{l=1}^L b_l$$

and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \dots\}$ for all k, l , and if

$$f(x) \geq 0$$

for all x , then

$$K + L \ll D^2 (\log D)^2.$$

Similarly, we answer a conjecture of A. Borisov [3] about the boundedness of such step functions.

Theorem 1.2. *If*

$$f(x) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor,$$

where

$$\sum_{k=1}^K a_k = \sum_{l=1}^L b_l$$

and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \dots\}$ for all k, l , and if

$$|f(x)| \leq A$$

for all x , then

$$K + L \ll A^2 (\log A)^2.$$

Remark 1.3. These theorems will be stated and proved in a more explicit form in Section 3, and some comments will be made about the achieved bounds in Section 4.

Concretely, Theorem 1.1 says that if $L + K$ is large in terms of $L - K$, then $f(x)$ must take negative values. Equivalently, this gives restrictions on when a factorial ratio sequence can be an integer sequence. Similarly, Theorem 1.2 says that if $L + K$ is large, then $f(x)$ takes on large values.

The interest in Theorems 1.1 and 1.2 comes from different areas of mathematics. On the one hand, an understanding of the nonnegativity of such step functions is equivalent to an understanding of the integrality of sequences of ratios of factorial products. Additionally, after a change of variables, these step functions are some of the simplest functions that show up in the Nyman–Beurling real variable reformulation of the Riemann hypothesis, so it is possible that their study may shed some light on the subject—in fact, Vasyunin [11] studied these functions from this perspective. F. Rodriguez-Villegas [7] also has studied the integrality of such step functions and relations to families of Calabi-Yau manifolds. Moreover, A. Borisov [3] has shown that in some ways the classification of integral factorial ratios is equivalent to the classification of cyclic quotient singularities having certain properties.

In fact, Borisov recasts Theorem 1.2 as the following statement about cyclic quotient singularities.

Proposition 1.4. *Suppose $a \geq 0$ is any real number. Then for all large enough $d \in \mathbb{N}$, for all but finitely many $(x_1, x_2, \dots, x_d) \in T^d$ that define a cyclic quotient singularity with Shukarov minimal log-discrepancy at least $d/2 - a$, for some pair of indices $1 \leq i < j \leq d$ we have $x_i + x_j = 1$.*

Proof. The appears as [3, Conjecture 3], where Borisov shows that it follows from Theorem 1.2. \square

Our theorems will actually be proved by giving a lower bound on the mean square of a step function of the form (4). To give our bound we will notice that the Fourier coefficients of such a function have some nice arithmetic properties. In fact, the Dirichlet series whose coefficients are the Fourier coefficients of $f(x)$ is just the product of the Riemann ζ -function and a Dirichlet polynomial, so, combining Parseval’s Theorem with a theorem of Carlson, we are instead able to study the appropriate Dirichlet series on a vertical line.

Our original version of this paper proved Theorems 1.1 and 1.2 with bounds that were exponential in $K - L$ and A . We are indebted to the K. Soundararajan for an argument (namely, the approximation of the ζ -function by a truncated Euler product) which leads to much improved lower bounds in Theorem 3.1 (and hence Theorems 1.1 and 1.2) and simplifies our exposition. After completing our original work we learned that E. Bombieri and J. Bourgain [2] also obtained our Theorem 1.2 through different methods.

1.1. Notation. $\lfloor x \rfloor$ denotes the *floor* of x , which is the largest integer less than or equal to x . $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x . Also $e(x) := \exp(2\pi i x) := e^{2\pi i x}$. γ denotes Euler’s constant

$$\gamma := \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log(n) \right) \approx 0.577215664901533.$$

1.2. Acknowledgements. We thank J. C. Lagarias for suggesting the problem that led to this paper and for helpful comments on earlier drafts. We also thank A. Borisov for some comments and for providing a preprint of [3], and E. Bombieri for providing a manuscript with an alternate proof of our theorems.

2. THE CONNECTION BETWEEN FACTORIAL RATIOS AND STEP FUNCTIONS

The connection that the step functions which we study have with factorial ratio sequences comes from the following theorem.

Proposition 2.1 (Landau [5]). *Let $a_{i,k}, b_{j,k} \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq l, 1 \leq j \leq k, 1 \leq k \leq r$ and let*

$$A_i(x_1, x_2, \dots, x_r) = \sum_{k=1}^r a_{i,k} x_k$$

and

$$B_j(x_1, x_2, \dots, x_r) = \sum_{k=1}^r b_{j,k} x_k.$$

(That is, A_i and B_j are linear forms in r variables with nonnegative integral coefficients.) Then the factorial ratio

$$\frac{\prod_{i=1}^l A_i(x_1, x_2, \dots, x_r)!}{\prod_{j=1}^k B_j(x_1, x_2, \dots, x_r)!}$$

is an integer for all $(x_1, \dots, x_r) \in \mathbb{Z}_{\geq 0}^r$ if and only if the function

$$F(y_1, \dots, y_r) = \sum_{i=1}^l \lfloor A_i(y_1, \dots, y_r) \rfloor - \sum_{j=1}^k \lfloor B_j(y_1, \dots, y_r) \rfloor$$

is nonnegative for all $(y_1, \dots, y_r) \in [0, 1]^r$.

A special case of this theorem is the following.

Corollary 2.2. *Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)! (a_2 n)! \cdots (a_K n)!}{(b_1 n)! (b_2 n)! \cdots (b_L n)!}$$

Then u_n is an integer for all n if and only if

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

is nonnegative for all x .

In [11] Vasyunin considered functions of this type taking on only the values 0 and 1. A simple generalization of a proposition of Vasyunin says that a necessary condition that such a function take values only in the range $0 \dots D$ is that $L - K = D$.

Lemma 2.3. *Suppose that $f(x)$ is a function of the form*

$$f(x) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

with $a_k, b_l \in \mathbb{Z}$, and that $f(x)$ is bounded. Then $\sum_{k=1}^K a_k = \sum_{l=1}^L b_l$ and, for any n , there exists some x such that $f(x) = -n$ if and only if there exists some x' such that $f(x') = L - K + n$. In particular, $f(x)$ is nonnegative if and only if the maximum value of f is $L - K > 0$.

Proof. The first assertion is clear, for if $\sum a_k \neq \sum b_l$, then $f(x)$ is unbounded. Now we know that $f(x)$ is periodic with period 1. Now, for any z that is not an integer we have $\lfloor z \rfloor + \lfloor -z \rfloor = -1$, so for any z for which none of $a_i z, b_j z$ is an integer, we have

$$f(z) + f(-z) = L - K,$$

from which the assertion follows. \square

3. PROOF OF THE MAIN THEOREM

We study functions of the form

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

with

$$a_k, b_l \in \mathbb{N}$$

subject to the conditions

$$(5) \quad \sum_{k=1}^K a_k = \sum_{l=1}^L b_l \text{ and } a_k \neq b_l \text{ for all } k, l.$$

The first condition ensures that $f(x; \mathbf{a}, \mathbf{b})$ is periodic with period 1 and the second is a nontriviality condition. Thus $f(x; \mathbf{a}, \mathbf{b})$ is a periodic integer valued step function that has possible jump discontinuities at points n/a_k or n/b_l for $n \in \mathbb{Z}$. Notice that, using the relation $\{x\} = x - \lfloor x \rfloor$ and the condition (5), $f(x; \mathbf{a}, \mathbf{b})$ can be rewritten in terms of combinations of fractional parts as

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{l=1}^L \{b_l x\} - \sum_{k=1}^K \{a_k x\},$$

which is the form that we will generally use. In this section, we will give the following explicit lower bound on the mean square of such step functions.

Theorem 3.1. *Let*

$$f(x) = \sum_{k=1}^K \lfloor a_k x \rfloor - \sum_{l=1}^L \lfloor b_l x \rfloor$$

where

$$\sum_{k=1}^K a_k = \sum_{l=1}^L b_l$$

and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \dots\}$ for all k, l , then for all $M > 285$

$$(6) \quad \int_0^1 f(x)^2 dx \geq \frac{(K-L)^2}{4} + \frac{(K+L)}{2\pi^2} \left[\frac{e^{-\gamma}}{(\log M)} \left(1 - \frac{1}{2(\log M)^2} \right) \right]^2 \\ - \frac{(K+L)^2}{2\pi^2} \left[\frac{2}{(M-1)^{1/2}} \left(\frac{e^{-\gamma}}{(\log M)} \left(1 - \frac{1}{2(\log M)^2} \right) \right)^{-1} + \frac{1}{M-1} \right].$$

Remark 3.2. In this theorem M is a parameter that we will be free to choose to optimize this lower bound. We obtain easily stated asymptotic results below by choosing $M = (K+L)^4$ and give examples of optimized bounds that can be obtained in Section 4. (The condition that $M > 285$ in the above statement comes from our use of Rosser and Schoenfeld's [8] explicit formulation of Merten's Theorem (see Lemma 3.8).)

Granting this theorem for a moment, we may now easily prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. Take M to be $(K+L)^4$. Then for fixed $K - L$, equation (6) gives

$$\max_{x \in \mathbb{R}} |f(x)|^2 \geq \int_0^1 f(x)^2 dx \gg \frac{(K+L)}{(\log(K+L))^2}.$$

Now let $N = K + L$ and $A = \max_{x \in \mathbb{R}} |f(x)|$. Then we have

$$\frac{N}{(\log N)^2} \ll A^2.$$

Additionally, since this implies that $A \gg N^{1/3}$, we may write

$$N \ll N \frac{(\log A)^2}{(\log N)^2} \ll A^2 (\log A)^2.$$

This gives Theorem 1.2. Theorem 1.1 follows from Lemma 2.3, which said that f is nonnegative if and only if the maximum value of f is $L - K$. \square

To prove Theorem 3.1, we begin with the Fourier expansion for our step functions.

Lemma 3.3. *Suppose that $f(x) = \sum_{l=1}^L \{b_l x\} - \sum_{k=1}^K \{a_k x\}$. Then the Fourier expansion of f is*

$$f(x) = \frac{L-K}{2} + \frac{1}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \left[\sum_{a_k | n} a_k - \sum_{b_l | n} b_l \right] e(nx).$$

That is, $\hat{f}(0) = (L-K)/2$ and for $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi i n} \left[\sum_{a_k | n} a_k - \sum_{b_l | n} b_l \right].$$

Proof. The Fourier expansion for the fractional part of x is

$$(7) \quad \{x\} = \frac{1}{2} - \frac{1}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e(nx)}{n}.$$

Using this we can write

$$\sum_{l=1}^L \{b_l x\} - \sum_{k=1}^K \{a_k x\} = \frac{L-K}{2} - \frac{1}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \left[\sum_{l=1}^L e(n b_l x) - \sum_{k=1}^K e(n a_k x) \right].$$

Changing the order of summation we get

$$\sum_{k=1}^K [a_k x] - \sum_{l=1}^L [b_l x] = \frac{L-K}{2} + \frac{1}{2\pi i} \left[\sum_{k=1}^K \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e(na_k x)}{n} - \sum_{l=1}^L \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e(nb_l x)}{n} \right].$$

Now extracting the coefficient of $e(mx)$ in the sum we see that for $m \neq 0$ we have

$$\hat{f}(m) = \frac{1}{2\pi i} \left[\sum_{\substack{n, a_k \\ na_k = m}} \frac{1}{n} - \sum_{\substack{n, b_l \\ nb_l = m}} \frac{1}{n} \right].$$

Now the result follows on replacing the n in the sum with $n = m/a_k$. \square

Remark 3.4. From now on we set $D = L - K$. It is convenient to subtract off the first Fourier coefficient of $f(x)$ and to consider

$$\int_0^1 \left| f(x) - \frac{D}{2} \right|^2 dx.$$

On considering the Fourier expansion, it is easy to see that

$$(8) \quad \int_0^1 |f(x)|^2 dx = \int_0^1 \left| f(x) - \frac{D}{2} \right|^2 dx + \frac{D^2}{4},$$

so it is simple to transfer a lower bound on one to a lower bound on the other.

Notice also that $|\hat{f}(n)| = |\hat{f}(-n)|$, so it follows from Parseval's theorem that

$$(9) \quad \int_0^1 |f(x) - D/2|^2 dx = 2 \sum_{n=1}^{\infty} |\hat{f}(n)|^2.$$

Remark 3.5. We now notice a Möbius inversion-type formula for the Fourier coefficients of $f(x) = f(x; \mathbf{a}, \mathbf{b})$. Let

$$(10) \quad g(n) = g(n; \mathbf{a}, \mathbf{b}) = \#\{a_k : a_k = n\} - \#\{b_l : b_l = n\}.$$

Then from Lemma 3.3 we see that for $n \geq 1$ we have

$$\hat{f}(n) = \frac{1}{2\pi i} \sum_{d|n} \frac{dg(d)}{n}.$$

Or, forming the Dirichlet series

$$G(s) = D(g, s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

(note that G is actually given by a finite sum) and

$$F(s) = D(\hat{f}, s) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{n^s},$$

we have the relation

$$(11) \quad G(s)\zeta(s+1) = 2\pi i F(s),$$

where $\zeta(s) = D(1, s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann ζ -function.

To estimate the mean square of f we use the following theorem of Carlson [4] to relate $\sum_{n=1}^{\infty} |\hat{f}(n)|^2$ to a mean value of $|G(it)|^2$.

Proposition 3.6. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in some half-plane. Then if $f(s)$ is analytic and of finite order for $\sigma \geq \alpha$, and*

$$\frac{1}{2T} \int_{-T}^T |f(\alpha + it)|^2 dt$$

is bounded as $T \rightarrow \infty$, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}.$$

Proof. See [10, Section 9.51]. □

It follows immediately from this proposition and the relation (11) that for all $\sigma > 0$ we have

$$(12) \quad \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n^{2\sigma}} = \lim_{T \rightarrow \infty} \int_{-T}^T |G(\sigma + it)\zeta(1 + \sigma + it)|^2 dt.$$

Eventually we will let σ tend to zero, but first we estimate the right hand side of (12) using a truncated Euler product for the ζ -function. Define the main term

$$\zeta_M(s) = \prod_{p \leq M} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and let $\zeta_R(s)$ be defined by $\zeta(s) = \zeta_M(s) + \zeta_R(s)$. We will use the following lemmas which give bounds on the size of ζ_M .

Lemma 3.7. *If $\operatorname{Re}(s) \geq 1$ then*

$$(13) \quad \frac{1}{\zeta_M(1)} \leq |\zeta_M(s)| \leq \zeta_M(1).$$

Proof. This is just a restatement of the fact that for all s with real part greater than 1, and all integers $n > 0$, we have

$$\left(1 - \frac{1}{n}\right) \leq \left|1 - \frac{1}{n^s}\right|^{-1} \leq \left(1 - \frac{1}{n}\right)^{-1}.$$

□

Lemma 3.8 (Effective Mertens' bound). *For all $M > 285$,*

$$(14) \quad \zeta_M(1) \leq \frac{\log M}{e^{-\gamma}} \left(1 - \frac{1}{2(\log M)^2}\right)^{-1}$$

Proof. See Rosser and Schoenfeld [8, Theorem 7]. □

Proof of Theorem 3.1. We now write

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |G(\sigma + it)\zeta(1 + \sigma + it)|^2 dt \\ = \frac{1}{2T} \int_{-T}^T |G(\sigma + it)\zeta_M(1 + \sigma + it)|^2 dt + E(M, T), \end{aligned}$$

where

$$E(M, T) = \frac{1}{2T} \int_{-T}^T |G(\sigma + it)| |\zeta_M(1 + \sigma + it) + \zeta_R(1 + \sigma + it)|^2 dt \\ - \frac{1}{2T} \int_{-T}^T |G(\sigma + it) \zeta_M(1 + \sigma + it)|^2 dt.$$

Applying the triangle inequality in the form $||A| - |B|| \leq |A - B|$ to the above, we obtain

$$|E(M, T)| \leq \frac{1}{2T} \int_{-T}^T |G(\sigma + it)|^2 |\zeta_R(1 + \sigma + it)|^2 dt \\ + \frac{1}{2T} \int_{-T}^T 2 |G(\sigma + it)^2 \zeta_M(1 + \sigma + it) \zeta_R(1 + \sigma + it)| dt.$$

On inserting the bound for ζ_M from Lemma 3.7, the bound $|G(\sigma + it)| \leq K + L$, and applying Cauchy's inequality to the integral $\int_{-T}^T |\zeta_R(1 + \sigma + it)| dt$, we have

$$|E(M, T)| \leq 2(K + L)^2 \zeta_M(1) \left[\frac{1}{2T} \int_{-T}^T |\zeta_R(1 + \sigma + it)|^2 dt \right]^{1/2} \\ + (K + L)^2 \frac{1}{2T} \int_{-T}^T |\zeta_R(1 + \sigma + it)|^2 dt.$$

Define $E(M) := \lim_{T \rightarrow \infty} E(M, T)$. Then

$$|E(M)| \leq 2(K + L)^2 \zeta_M(1) \left[\sum_{n \geq M} \frac{1}{n^{2+2\sigma}} \right]^{1/2} + (K + L)^2 \sum_{n \geq M} \frac{1}{n^{2+2\sigma}} \\ \leq 2(K + L)^2 \zeta_M(1) \left[\sum_{n \geq M} \frac{1}{n^2} \right]^{1/2} + (K + L)^2 \sum_{n \geq M} \frac{1}{n^2}.$$

Inserting the bound

$$\sum_{n \geq M} \frac{1}{n^2} \leq \frac{1}{M-1},$$

we get

$$|E(M)| \leq \frac{2(K + L)^2 \zeta_M(1)}{(M-1)^{1/2}} + \frac{(K + L)^2}{M-1}.$$

Now,

$$\frac{1}{2T} \int_{-T}^T |F(\sigma + it)|^2 dt = \frac{1}{(2\pi)^2} \frac{1}{2T} \int_{-T}^T |G(\sigma + it) \zeta_M(1 + \sigma + it)|^2 dt + \frac{E(M, T)}{4\pi^2},$$

so inserting the bound from our Lemma 3.7 and taking the limit as $T \rightarrow \infty$ we get

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \geq \frac{1}{\zeta_M(1)^2} \frac{1}{(2\pi)^2} \sum_{n=1}^{\infty} |g(n)|^2 + \frac{E(M)}{4\pi^2} \\ \geq \frac{1}{\zeta_M(1)^2} \frac{1}{(2\pi)^2} \sum_{n=1}^{\infty} |g(n)|^2 - \frac{|E(M)|}{4\pi^2}.$$

Putting in the bounds for $E(M)$, we get

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \geq \frac{1}{\zeta_M(1)^2} \frac{1}{(2\pi)^2} (K+L) - \frac{(K+L)^2}{4\pi^2} \left[\frac{2\zeta_M(1)}{(M-1)^{1/2}} + \frac{1}{M-1} \right].$$

And finally, putting in the bounds on $\zeta_M(1)$, we get

$$(15) \quad \sum_{n=1}^{\infty} |\hat{f}(n)|^2 \geq \frac{(K+L)}{4\pi^2} \left[\frac{e^{-\gamma}}{(\log M)} \left(1 - \frac{1}{2(\log M)^2} \right) \right]^2 - \frac{(K+L)^2}{4\pi^2} \left[\frac{2}{(M-1)^{1/2}} \left(\frac{e^{-\gamma}}{(\log M)} \left(1 - \frac{1}{2(\log M)^2} \right) \right)^{-1} + \frac{1}{M-1} \right].$$

From this we obtain the following explicit lower bound on the mean square of f . The theorem now follows immediately from (15) and fact that

$$\int_0^1 f(x)^2 dx = 2 \sum_{n=1}^{\infty} |\hat{f}(n)|^2 + \frac{(K-L)^2}{4}.$$

□

4. EXPLICIT BOUNDS AND REMARKS

Let $\mathbf{B}(D)$ denote the maximal number of terms that a step function of the form

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^K [a_k x] - \sum_{l=1}^L [b_l x]$$

may have if $L - K = D$, $a_i, b_j \in \mathbb{N}$, $\sum a_i = \sum b_j$ and $a_i \neq b_j$ for all i, j , and $f(x) \geq 0$ for all x . With this notation, Theorem 1.1 says that $\mathbf{B}(D)$ is finite, and that, moreover $\mathbf{B}(D) \ll D^2 \log D$. Using equation 15 we may compute some explicit bounds on $\mathbf{B}(D)$.

For example, when $D = 1$ such a function $f(x)$ that only takes on the values 0 and 1 must have $\sum_{n=1}^{\infty} |\hat{f}(n)|^2 = \frac{1}{4}$. If $K + L = 112371$ and $M = 112371^{4.96215}$, the right side of 15 is ≈ 0.250000802 . Additionally, the right side of (15) is an increasing function of $K + L$ when M is set to be a fixed power of $K + L$, so it follows that $\mathbf{B}(1) < 112371$.

Similarly we obtain $\mathbf{B}(2) < 502827$ by taking $L + K = 502827$ and $M = 502827^{4.6602}$, in which case the right side of (15) is ≈ 1.00000138 , but it must be the case that $\sum_{n=1}^{\infty} |\hat{f}(b)|^2 \leq 1$ if $f(x)$ takes values in precisely in the range $\{0, 1, 2\}$.

These bounds are far from best possible. In fact it is known that $\mathbf{B}(1) = 9$ (see [1], which implies that $\mathbf{B}(D) \geq 9D$ for all D). Computations by the second author suggest that it might be the case that $\mathbf{B}(2) = 18$. Additionally, for the case of $D = 3$, a direct computer search by the second author has checked that there are no functions with 29 terms that take values only in the range $\{0, 1, 2, 3\}$ when $\sum a_i = \sum b_j \leq 60$. (However, the search space would have to be greatly enlarged for this to be truly convincing evidence that $\mathbf{B}(3) = 27$.)

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